A Hilbert space approach to cognitive psychology

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Basic idea

Let the random variable X represent the decision of concern.

Here, $X = x_i$ with the probability p_i.

Let ϵ, with density $f(y)$, represent noise that obscures the finding of X.

Then a noisy observation of X is represented by the detection of

$$\xi = X + \epsilon \, .$$

After detection, the *a priori* probability p_i changes into the *a posteriori* probability $\pi_i(\xi)$:

$$p_i \to \pi_i(\xi) = \frac{p_i f(\xi - x_i)}{\sum_j p_j f(\xi - x_j)} \, .$$

This is just the Bayes formula.

A Hilbert space reformulation

Let the random variable X will be represented on Hilbert space $\mathcal{H}$ by a matrix $\hat{X}$, with $(x_i, |x_i\rangle)$ its eigenvalues and eigenvectors.

The uncertainties about $\hat{X}$ is represented by a pure state

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{p_i}|x_i\rangle.$$

The noise will be represented by $\hat{\epsilon}$, and the state of the noise is $|\eta\rangle = \sqrt{f(y)}$.

The state of the total system initially is thus a product state $|\psi\rangle\,|\eta\rangle$.

Acquisition of partial information can be modelled by observing

$$\hat{\xi} = \hat{X} + \hat{\varepsilon}.$$

The transformation of the state after measurement is given by the von Neumann-Lüders projection postulate.

Writing

$$\hat{\Pi}_\xi = \sum_k \delta(y - \xi + x_k)|x_k\rangle\langle x_k|,$$

the state of the system after information acquisition is

$$\hat{\rho}_\xi = \frac{\hat{\Pi}_\xi|\psi\rangle|\eta\rangle\langle\psi|\langle\eta|\hat{\Pi}_\xi}{\mathrm{tr}\left(\hat{\Pi}_\xi|\psi\rangle|\eta\rangle\langle\psi|\langle\eta|\hat{\Pi}_\xi\right)}.$$

This density matrix is a projection operator onto a pure state

$$|\psi(\xi)\rangle = \sum_k \sqrt{\pi_k(\xi)}\,|x_k\rangle,$$

where

$$\pi_k(\xi) = \frac{p_k\, f(\xi - x_k)}{\sum_m p_m f(\xi - x_m)}.$$

Thus, the von Neumann-Lüders projection postulate implies the Bayes formula.

Meaning of the von Neumann-Lüders-Bayes postulate

(i) Of all the inferences consistent with the observation ξ, the Lüders state $|\psi(\xi)\rangle$ is the 'closest' to the prior state $|\psi\rangle$:

$$\begin{array}{rcccccc} (X,\epsilon) & = & (x_1,\xi-x_1) & (x_2,\xi-x_2) & (x_3,\xi-x_3) & \cdots & (x_N,\xi-x_N) \\ q_i & = & 1 & 0 & 0 & \cdots & 0 \\ q_i & = & q & 1-q & 0 & \cdots & 0 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q_i & = & q_1 & q_2 & q_3 & \cdots & q_N \end{array}$$

(ii) The change of the state $|\psi\rangle \to |\psi(\xi)\rangle$ on average minimises the uncertainty in $\hat{X}$.

(iii) The change is induced only by the arrival of new information.

Information acquisition leads to a decoherence effect

A psychologist wishing to predict the statistics of the behaviour of a person who has acquired information relevant to decision making will *a priori* not know the observed value of ξ.

Hence in this case the density matrix $\hat{\rho}_\xi$ has to be averaged over ξ:

$$\mathbb{E}[\hat{\rho}_\xi] = \sum_{k,l} \sqrt{p_k p_l}\, \Lambda(\omega_{kl})\, |x_k\rangle\langle x_l|,$$

where $\omega_{kl} = x_k - x_l$ and

$$\Lambda(\omega) = \int_{-\infty}^{\infty} \sqrt{f(y)}\, \sqrt{f(y-\omega)}\, \mathrm{d}y.$$

Evidently, $0 \leq \Lambda(\omega_{kl}) \leq 1$ and $\Lambda(\omega_{kk}) = 1$ for all k, l.

Because the initial state of mind $|\psi\rangle\langle\psi|$ in this basis has the matrix elements $\{\sqrt{p_k p_l}\}$, it follows that an external observer (e.g., a psychologist) will perceive a decoherence effect.

Conversely decoherence implies information acquisition

It seems reasonable to conjecture that conversely, decoherence implies that the system has on average acquired information (Brody & Trewavas 2023).

While it appears difficult to prove this in general, it is possible to prove the conjecture in two-dimensional Hilbert spaces.

Sequential updating

In most cases, information arrives in the form of a time series.

A simple such time series would be

$$\hat{\xi}_t = \hat{X}t + \hat{\varepsilon}_t \,,$$

where the noise term $\hat{\varepsilon}_t$ is modelled by a standard Brownian motion $\{B_t\}$.

In this example, over a small time increment $\mathrm{d}t$ the initial state $|\psi\rangle|\eta\rangle$ is projected to the Lüders state $\propto \hat{\Pi}_{\xi_{\mathrm{d}t}}|\psi\rangle|\eta\rangle$.

By sequentially applying the projection operator for each time step and taking the limit, we obtain the state at time t:

$$|\psi(t,\xi_t)\rangle = \frac{1}{\sqrt{\Phi_t}} \sum_k \sqrt{p_k}\, \mathrm{e}^{\frac{1}{2}x_k\xi_t - \frac{1}{4}x_k^2 t}|x_k\rangle,$$

where $\xi_t = Xt + B_t$ and $\Phi_t = \sum_k p_k \mathrm{e}^{k_k\xi_t - \frac{1}{2}x_k^2 t}$ gives the normalisation.

Implied dynamics

Having obtained how the state of mind $|\psi_t\rangle = |\psi(t, \xi_t)\rangle$ evolves in time, we can work out its dynamics to find

$$\mathrm{d}|\psi_t\rangle = -\frac{1}{8}(\hat{X} - \langle \hat{X} \rangle_t)^2 |\psi_t\rangle \, \mathrm{d}t + \frac{1}{2}(\hat{X} - \langle \hat{X} \rangle_t)|\psi_t\rangle \, \mathrm{d}W_t,$$

where $\langle \hat{X} \rangle_t = \langle \psi_t | \hat{X} | \psi_t \rangle / \langle \psi_t | \psi_t \rangle$ and where

$$\mathrm{d}W_t = \mathrm{d}\xi_t - \langle \hat{X} \rangle_t \, \mathrm{d}t.$$

The process $\{W_t\}$ defined in this way is in fact a standard Brownian motion, known as the innovations process.

The innovations process represents the arrival of new information.

As for an external psychologist, the dynamics of the state $\hat{\rho}_t = \mathbb{E}[\hat{\rho}_{\xi_t}]$ is given by

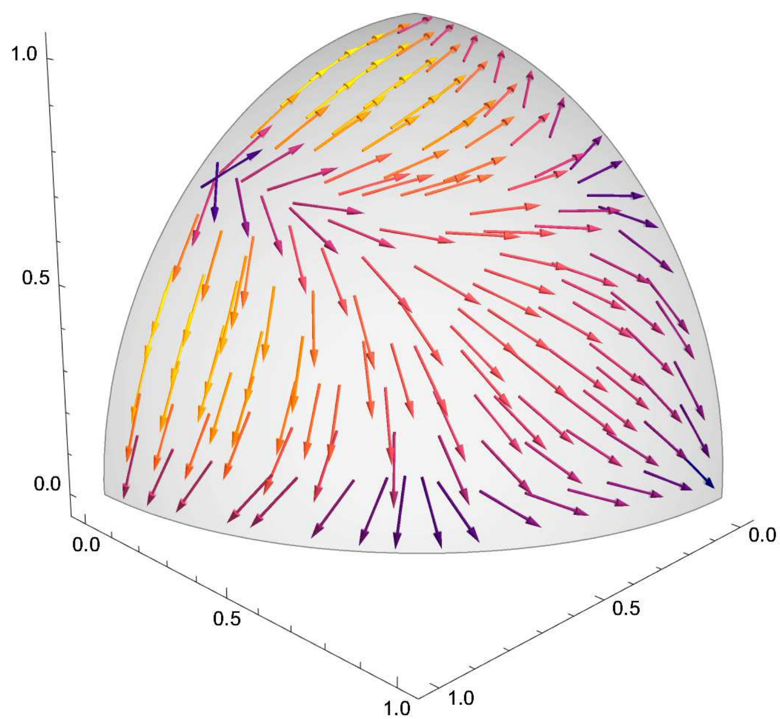
$$\frac{\partial \hat{\rho}_t}{\partial t} = \hat{X} \hat{\rho}_t \hat{X} - \frac{1}{2}\left(\hat{X}^2 \hat{\rho}_t + \hat{\rho}_t \hat{X}^2 \right).$$

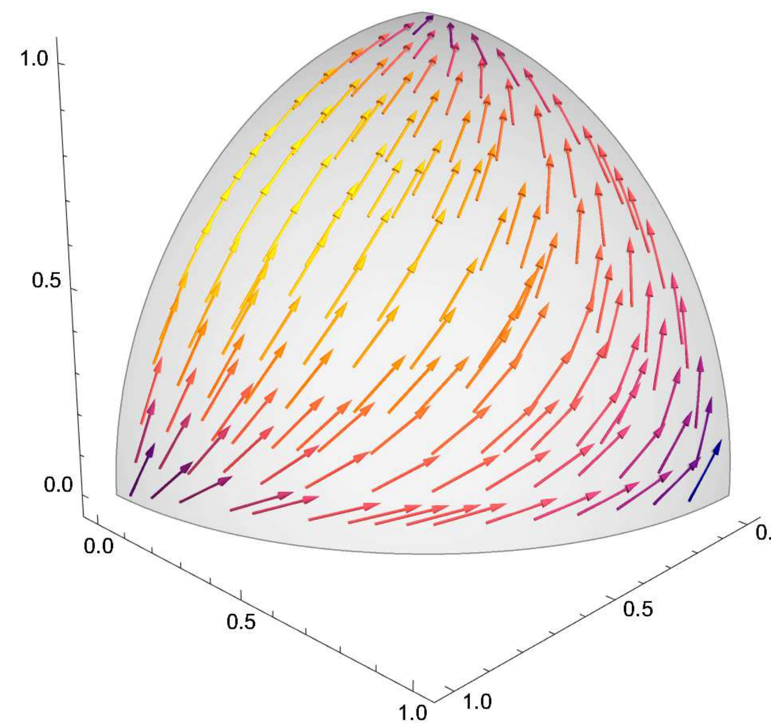
Projecting down the dynamics

On projective Hilbert space, the dynamical equation can be written in the form

$$\mathrm{d}\psi^a = -\frac{1}{16}\nabla^a V_X \,\mathrm{d}t + \frac{1}{4}\nabla^a X \,\mathrm{d}W_t,$$

where $V_X(\psi) = \langle\psi|(\hat{X} - X(\psi))^2|\psi\rangle/\langle\psi|\psi\rangle$ and $X(\psi) = \langle\psi|\hat{X}|\psi\rangle/\langle\psi|\psi\rangle$.





Limitation of classical reasoning

There are empirical examples in behavioural psychology that strongly indicate that not all decisions or opinions are compatible.

The issue with the classical updating of likelihoods based on the Bayes formula is that it is not well suited to characterise changes of the contexts.

In contrast, the von Neumann-Lüders postulate does not suffer from this issue, thus enables to explain many cognitive behaviours.

To illustrate the idea, consider the disjunction effect: One will do x given A occurs and will do x given A does not occur; yet will not do x when the outcome of A is unknown.

To illustrate how quantum formalism can describe such an effect, let the state of mind be given by

$$|\psi(\theta)\rangle = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}.$$

Let a typical 'yes-no' question on Hilbert space $\mathcal{H}$ by

$$\hat{Z} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A binary 'question' like $\hat{Z}$ will have two eigenvalues $+1$ and -1, corresponding to the 'yes' and 'no' answers.

The corresponding eivenvectors

$$|Z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |Z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

represent definite states of yes and no.

Given an initial state $|\psi(\theta)\rangle$ the probability of getting yes or no answers are:

$$|\langle\psi(\theta)|Z\pm\rangle|^2 = \left\{ \begin{array}{l} \cos^2 \frac{1}{2}\theta \\ \sin^2 \frac{1}{2}\theta \end{array} \right. .$$

But there can be an incompatible 'yes-no' question represented by

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Like $\hat{Z}$, this matrix has the eigenvalues ± 1 corresponding to yes-no answers.

However, the corresponding eigenvectors are now:

$$|X+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |X-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The von Neumann rules remain says the probabilities of getting the yes and no answers are

$$|\langle\psi(\theta)|X\pm\rangle|^2 = \begin{cases} \frac{1}{2}(1+\sin\theta) \\ \frac{1}{2}(1-\sin\theta) \end{cases}$$

Consider the 'exam vs holiday' example of Tversky and Shafir.

For an illustration, let $\theta = 2\pi/3$.

Then we have the values

$$\mathbb{P}(Z+) = \cos^2\frac{1}{2}\theta = 25\% \quad \text{and} \quad \mathbb{P}(Z-) = \sin^2\frac{1}{2}\theta = 75\%$$

for the likelihoods of the outcomes for $\hat{Z}$.

For the outcomes of $\hat{X}$ the likelihoods are

$$\mathbb{P}(X+) = \frac{1}{2}(1+\sin\theta) \approx 93\% \quad \text{and} \quad \mathbb{P}(X-) = \frac{1}{2}(1-\sin\theta) \approx 7\%.$$

Suppose that $\hat{Z}$ corresponds to the question "would you go for a holiday?"

The outcome of $\hat{X}$ tells whether you passed the exam or not.

If the outcome of $\hat{X}$ is **not known**, then the state of mind is given by $|\psi(\theta)\rangle$, so the likelihood of students purchasing a nonrefundable holiday is 25%.

However, if the outcome of the exam $\hat{X}$ is **known**, then the state of mind is either a pass state $|X+\rangle$ (which happens to 93% of the students) or a fail state $|X-\rangle$ (which happens to 7% of the students).

If the outcome is a pass, then the state of mind transforms from $|\psi(\theta)\rangle$ to $|X+\rangle$ in accordance with the von Neumann postulate.

Then conditional on this outcome, the likelihood of saying yes to the holiday question is given by $|\langle X+|Z+\rangle|^2 = 50\%$.

Similarly, if the exam outcome is a fail, then the state transforms as $|\psi(\theta)\rangle \to |X-\rangle$.

In this case, the likelihood of saying yes to the holiday question is given by $|\langle X - |Z+\rangle|^2 = 50\%$.

Summing up, if the exam outcome is unknown, the likelihood of students buying the holiday is 25%; whereas if the exam result is known, the likelihood changes to 50%, irrespective of the exam outcome.

Possible implication in artificial intelligence

Let us accept the idea that probability assignment rules of quantum theory can describe human thinking more adequately than their classical counterparts.

Then it follows that machine learning tools (e.g., ANN, Hyperdimensional computing, etc.) that are based on classical probability will ultimately fail to replicate human behaviours.

Instead, an **artificial quantum intelligence** implementing quantum probability rules of von Neumann and Lüders can be envisaged that will more accurately replicate human logic.

For such an architecture to be useful, more research is needed to uncover the meaning and the implication of incompatible decisions in cognitive psychology.

References

1. D. C. Brody (2022) Noise, fake news, and tenacious Bayesians. *Frontiers in Psychology* **13**, 797904

2. D. C. Brody (2022) Open quantum dynamics for plant motions. *Scientific Reports* **12**, 3042

3. D. C. Brody & A. J. Trewavas (2023) Biological efficiency in processing information in green plants. (arXiv:2209.11054)

4. D. C. Brody & T. Yuasa (2023) Three candidate election strategy. (arXiv:2305.00693)

5. D. C. Brody (2023) Quantum formalism for cognitive psychology. (arXiv:2303.06055)